Semi-infinite cohomology in conformal field theory and 2D gravity

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We discuss various techniques for computing the semi-infinite cohomology of highest weight modules which arise in the BRST quantization of two dimensional field theories. In particular, we concentrate on two such theories – the G/H coset models and 2D gravity coupled to $c \leq 1$ conformal matter.

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1. Introduction

Among the interesting new phenomena discovered in recent studies of two dimensional gravity models is the existence of so-called discrete states [1,2] in the physical spectrum. Models with a $c \leq 1$ conformal matter sector have been extensively studied in the framework of BRST quantization, where the physical states represent nontrivial cohomology classes of the BRST charge – the discrete states are then classes present at only a countable set of momenta (see refs. [3–8] and references therein). Perhaps even more striking is the fact that they occur over a range of different ghosts numbers, quite unlike the physical spectrum of critical string theories studied previously.

Heuristically, this novel behaviour for the BRST cohomology arises from two distinct sources: first is the obvious difference in that the world-sheet scalar fields have background charges for the non-critical models (the corresponding Fock space is called a Feigin–Fuchs module); second, in the case of $c < 1$ matter, one must project from the Fock spaces onto the irreducible modules of the Virasoro algebra which comprise the spectrum of such models. This projection can be efficiently carried out [5,6] using free field resolutions of the irreducible module in terms of Feigin–Fuchs modules [9]. Thus, both sources are somewhat related, and in all cases the problem of identifying the physical states of the theory can be reduced to the computation of the BRST cohomology of a tensor product of two Feigin–Fuchs modules.

Mathematically, the BRST cohomology discussed above is an example of semi-infinite cohomology [10] of the Virasoro algebra. Another class of algebras whose semi-infinite cohomology has been well discussed is the affine Kac–Moody algebras [10,11]. It is natural, then, to look for the physical setting of the corresponding BRST complex by analogy with that in 2D gravity. An immediate example is in the context of the representation theory of affine Kac–Moody algebras, a problem which is directly relevant to treatments of general G/H coset models [12,13]. In fact, when formulated as gauged Wess-Zumino-Witten models [14,15], the G/H coset theories have a natural nilpotent BRST operator [16] acting on the tensor product of highest weight modules of the current algebra of G and H, respectively. One of the main themes of this paper will be to analyze this complex, and, in particular to prove that its cohomology indeed yields the correct spectrum of states for the coset theory.

The BRST formulation of coset theories is a fairly old problem. In the case when H is Abelian, its cohomology was computed already in the original paper ref. [16]. For a general G/H theory, another complex was proposed in ref. [17], which was based on the free field realization of the WZW model with group G. Although it was possible in this formulation to compute the cohomology explicitly, and thus derive e.g. branching functions for arbitrary coset theories, the construction could not be recast in a covariant formulation from the point of view of conformal field theory. So a complete treatment of the correlation functions for these models was not possible. The extreme case where the complete group is gauged away, G/G coset models, defines a topological field theory. It was recently observed in this simpler setting that a suitable choice of the module describing the states of the H ($= G$) sector of the theory allows just such a covariant description, as well as an explicit computation of the BRST cohomology [18–22], in very much the same manner as in 2D gravity. One of the results we present here is a general discussion of the BRST cohomology for an arbitrary G/H coset theory. We also construct the free field resolutions for these coset models, and demonstrate how the corresponding extended BRST complex can be reduced to our previous formulation [17].

As is noted several times above, the computations here appear superficially similar to those in 2D gravity. Indeed there exist free field realizations of affine Kac–Moody algebras, and all physically interesting irreducible highest weight modules admit resolutions in terms of (twisted) Wakimoto modules [23–27], which take the role played by Feigin–Fuchs modules in the case of the Virasoro algebra. Thus, one may first establish the BRST cohomology of a tensor product of two (oppositely twisted) Wakimoto modules, and then project onto the irreducible module using a suitable resolution. However, it will become apparent that the two contexts are really quite different, the difference being due to a rather subtle property of Wakimoto modules; namely, that they are essentially defined by their semi-infinite cohomology with respect to a (twisted) maximal nilpotent subalgebra of the affine Kac–Moody algebra [23]. We will show that in fact a natural modification of standard results on semi-infinite cohomology of Lie algebras [11,28] allows a straightforward computation of the BRST cohomology for general coset models without ever resorting to an explicit realization of the Wakimoto modules! This should be compared to the situation in 2D gravity, where an analogous cohomological characterization of Feigin–Fuchs modules does not exist unless they are isomorphic with the Verma module or its dual. This forces one to be more resourceful, by either trying to exploit the explicit structure of the BRST operator [5], or by using the inherent $SO(2, \mathbb{C})$ symmetry of the problem to reduce the computation so that it can be treated by standard methods [6].

The overall structure of this paper is that we first survey some techniques of homological algebra, and then show how to use them in the computation of the BRST cohomology of coset models and 2D gravity. After introducing the basic objects of semi-infinite cohomology in section 2, we review standard computational techniques in section 3. They all utilize one kind or other of spectral sequence. The new result here is a generalization of the reduction theorem [11,28]. In section 4 we discuss cohomological definitions of Verma modules and Wakimoto modules of an affine Kac–Moody algebra, and summarize the corresponding resolutions of irreducible highest weight modules. With this machinery in hand, section 5 is devoted to analysing the BRST formulation of the coset models. In section 6 we compare it with that of 2D gravity, and present a simplified derivation of the discrete states in $c < 1$ models.

2. Definitions and conventions

For a Lie algebra g and a module V, the standard Lie algebra cohomology of g with values in V (see e.g. [29,30]) is simply the cohomology of the operator

$$d = \sum_A c^A [\pi(e_A) + \tfrac{1}{2}\pi_{\text{gh}}(e_A)], \qquad (2.1)$$

in the complex $C(\boldsymbol{g}, V)$, where

$$C(\boldsymbol{g}, V) = \bigoplus_{n \geq 0} C^n(\boldsymbol{g}, V), \qquad C^n(\boldsymbol{g}, V) = V \otimes \bigwedge^n \boldsymbol{g}^*. \qquad (2.2)$$

Here e_A, $A = 1, \cdots, \dim \boldsymbol{g}$, are the generators of $\boldsymbol{g}$, which act on V in the representation $\pi(\cdot)$, and on $\bigwedge^n \boldsymbol{g}^*$, the nth exterior power of the (restricted) dual $\boldsymbol{g}^*$, by $\pi_{\mathrm{gh}}(\cdot)$, which is induced from the coadjoint action of $\boldsymbol{g}$ on $\boldsymbol{g}^*$. The ghost operators $c^A = c(e^{A*})$ can be identified with the basis $\{e^{A*}\}$ in $\boldsymbol{g}^*$, dual to the generators e_A. It is convenient to introduce antighost operators, $b_A = b(e_A)$, which are canonically conjugate to c^A, i.e. $\{b_A, c^B\} = \delta^B_A$. Then we may identify $\pi_{\mathrm{gh}}(e_A) = -\sum_{B,C} c^B b_C f_{AB}{}^C$, where $f_{AB}{}^C$ are the structure constants of $\boldsymbol{g}$. One may notice that together the ghosts c^A and antighosts b_A span the Clifford algebra of $\boldsymbol{g} \oplus \boldsymbol{g}^*$, with a canonical pairing $\langle x, y^* \rangle = y^*(x)$. One can identify $\bigwedge^n \boldsymbol{g}^*$ with the subspace of the ghost Fock space with ghost number n. In this Fock space all c^A act as creation operators, and the vacuum, to which we assign ghost number (order) zero, is annihilated by all b_A. The differential d is nilpotent by virtue of the Jacobi identities. We will denote the resulting cohomology of order n by $H^n(\boldsymbol{g}, V)$.

There is an obvious difficulty when applying this construction to infinite dimensional algebras, as the differential (2.1) becomes a series in which an infinite number of terms may act nontrivially on a given state in the complex. To circumvent this problem, Feigin [10] proposed to replace the space of forms $\bigwedge^\bullet \boldsymbol{g}^*$ by a suitably defined space of semi-infinite forms $\bigwedge^{\infty/2+\bullet} \boldsymbol{g}^*$. As will become clear in a moment, his construction has its origin in the physicists treatment of the infinite negative energy problem in the quantization of fermion fields.

To define the semi-infinite cohomology we will restrict both the possible algebras and the possible modules. The first restriction is to $\mathbb{Z}$-graded Lie algebras. Any such $\boldsymbol{g} = \bigoplus_{i \in \mathbb{Z}} \boldsymbol{g}_i$ can be decomposed as

$$\boldsymbol{g} = \boldsymbol{n}_- \oplus \boldsymbol{t} \oplus \boldsymbol{n}_+, \qquad (2.3)$$

where

$$\boldsymbol{n}_- = \bigoplus_{i<0} \boldsymbol{g}_i, \quad \boldsymbol{t} = \boldsymbol{g}_0, \quad \boldsymbol{n}_- = \bigoplus_{i>0} \boldsymbol{g}_i. \qquad (2.4)$$

Corresponding to (2.3), the space of semi-infinite forms $\bigwedge^{\infty/2+\bullet} \boldsymbol{g}^*$ is defined as the ghost Fock space $\mathcal{F}_{\mathrm{gh}}$, which is generated by the ghost and antighost operators acting on a vacuum state ω_0 satisfying

$$b(x)\omega_0 \;=\; 0, \quad \text{for } x \in \boldsymbol{t} \oplus \boldsymbol{n}_+, \qquad (2.5)$$

$$c(y^*)\omega_0 \;=\; 0, \quad \text{for } y^* \in \boldsymbol{n}_-^* = (\boldsymbol{t} \oplus \boldsymbol{n}_+)^\perp. \qquad (2.6)$$

We assign to ω_0 the ghost number equal zero, and will refer to it as a "physical vacuum". One should note that, unlike previously, there are states with both positive as well as negative ghost numbers, and this will lead to two-sided complexes later on.

To make this discussion more concrete, let us assume g is one of the following:
 - a finite dimensional Lie algebra, $\mathbf{g}$,
 - an (untwisted) affine Kac–Moody algebra, $\widehat{\mathbf{g}}$ ($\widehat{\mathbf{g}} = L\mathbf{g} \oplus \mathbb{C}k \oplus \mathbb{C}d$ is the centrally extended loop algebra of $\mathbf{g}$),
 - the Virasoro algebra Vir.

We will denote by Δ_+ and Δ_- the space of positive roots and negative roots of $\mathbf{g}$, respectively, and by $\Delta = \Delta_+ \cup \Delta_-$ the total root space. Similarly for $\widehat{\mathbf{g}}$ we have $\widehat{\Delta}$, $\widehat{\Delta}_+$ and $\widehat{\Delta}_-$. We recall that in terms of finite dimensional roots we can express the affine roots as follows ($\widehat{\Delta} = \widehat{\Delta}_+ \cup \widehat{\Delta}_-$):

$$\widehat{\Delta}_+ = \{\alpha + n\delta \mid \alpha \in \Delta_+, \ n \geq 0\} \cup \{\alpha + n\delta \mid \alpha \in \Delta_-, \ n > 0\} \cup \{n\delta \mid n > 0\}, \quad (2.7)$$

where δ is the (imaginary) root dual to $d = -L_0$. Further, we will denote by P (resp. P_+) the space of integral weights (resp. integral dominant weights) of $\mathbf{g}$, and by P^k (resp. P^k_+) the space of affine integral weights (affine integrable weights) of $\widehat{\mathbf{g}}$ at level k. We recall that an affine weight $\widehat{\lambda} \in P^k$ can be represented by its finite dimensional projection and the level k as $\widehat{\lambda} = \lambda + k\Lambda_0$, where Λ_0 is dual to k. For convenience we will usually use this finite dimensional parametrization. We will also denote by W the Weyl group of $\mathbf{g}$, and by $\widehat{W}$ that of $\widehat{\mathbf{g}}$. The latter is isomorphic to the semi-direct product $\widehat{W} = W \times T$, where T is the long-root lattice, such that for $\widehat{w} = t_\gamma w$, $\widehat{w}\lambda = w\lambda + k\gamma$ [31].

The $\mathbb{Z}$-grading for $\mathbf{g}$ and $\widehat{\mathbf{g}}$, which will be denoted by deg $(\cdot)$, is defined by the height function in the root space. Explicitly, for a finite dimensional $\mathbf{g}$ we have

$$\deg(t) = 0, \quad \deg(\mathbf{g}_\alpha) = \mathrm{ht}(\alpha) = (\rho, \alpha), \quad \alpha \in \Delta, \quad (2.8)$$

where ρ is the element of t^* such that $(\rho, \alpha_i^\vee) = 1$ for the simple roots α_i, $i = 1, \ldots, \ell = \mathrm{rank}\,\mathbf{g}$. Then (2.3) is simply the Cartan decomposition of $\mathbf{g}$,

$$\mathbf{g} = \mathbf{n}_- \oplus t \oplus \mathbf{n}_+ = \left(\bigoplus_{\alpha \in \Delta_-} \mathbf{g}_\alpha\right) \oplus t \oplus \left(\bigoplus_{\alpha \in \Delta_+} \mathbf{g}_\alpha\right). \quad (2.9)$$

A similar decomposition holds for the affine Kac–Moody algebra $\widehat{\mathbf{g}}$, if we take $\widehat{\rho} = \rho + h^\vee \Lambda_0$, where $h^\vee$ is the dual Coxeter number of $\mathbf{g}$.

In Vir the grading is defined by the eigenvalues of $-L_0$. In all three cases, the zero-degree subalgebra is Abelian, and the space of semi-infinite forms has a weight space decomposition with respect to it. All of the above carries over to subalgebras of any of these three algebras.

The second restriction is placed on the possible modules, we restrict the class of g-modules to the category $\mathcal{O}$ [32,31].

Definition 2.1. A module V is in the category $\mathcal{O}$ if

(i) V has a weight space decomposition $V = \bigoplus_{\lambda \in P(V)} V_\lambda$, with finite dimensional weight spaces V_λ.

(ii) There exists a set of weights $\lambda_1, \cdots, \lambda_n$ such that $P(V) \subset \bigcup_{i=1}^{n} D(\lambda_i)$, where $D(\lambda)$ denotes the set of all descendant weights μ of λ, $D(\lambda) = \{\mu \in P \mid \lambda - \mu \in \Delta_+\}$.

If $\boldsymbol{g} = \boldsymbol{a}$ is some subalgebra of the above three algebras ($\boldsymbol{g}, \widehat{\boldsymbol{g}}$ or Vir), we will consider only such $\boldsymbol{a}$-modules V which are also $\boldsymbol{t}$-modules, with the weight space decomposition as above. A simple consequence of this definition is that V is locally $\boldsymbol{n}_+$-finite [1], i.e. for any $v \in V$ the subspace $\mathcal{U}(\boldsymbol{n}_+)v$ is finite dimensional, where $\mathcal{U}(\boldsymbol{n}_+)$ is the enveloping algebra of $\boldsymbol{n}_+$. This category $\mathcal{O}$ of $\boldsymbol{g}$ modules is rather large – it includes Verma modules, Feigin–Fuchs and Wakimoto modules, their tensor products, submodules, quotients, duals, etc.

For a module V in the category $\mathcal{O}$ we set

$$C^{\infty/2+\bullet}(\boldsymbol{g}, V) = V \otimes \bigwedge^{\infty/2+\bullet} \boldsymbol{g}^*, \tag{2.10}$$

and consider an operator $d: C^{\infty/2+n}(\boldsymbol{g}, V) \to C^{\infty/2+n+1}(\boldsymbol{g}, V)$,

$$d = \sum_A c^A \pi(e_A) - \tfrac{1}{2}: \sum_{A,B,C} c^A c^B b_C f_{AB}{}^C: + c(\beta), \tag{2.11}$$

where : : denotes the normal ordering with respect to ω_0, and β is some constant element of $\boldsymbol{g}^*$. Since only a finite number of terms contribute to the action of d on any given state, this operator is certainly well defined. The following condition for the nilpotency of d is a classic result (see e.g. ref. [10,11]):

Theorem 2.1. If the total central charge of the representation

$$\Pi(e_A) = \{d, b_A\} = \pi(e_A) - : \sum_{B,C} c^B b_C f_{AB}{}^C :, \tag{2.12}$$

of $\boldsymbol{g}$ ($\boldsymbol{g} = \boldsymbol{g}, \widehat{\boldsymbol{g}}$ or Vir) in $C^{\infty/2+\bullet}(\boldsymbol{g}, V)$ vanishes, then there exists a β such that d defined in (2.11) is nilpotent, i.e. $d^2 = 0$.

In fact, for the algebras considered in this paper, we may always set β to zero by modifying the normal ordering prescription, e.g. by ordering with respect to the $SL(2,\mathbb{R})$ invariant vacuum of conformal field theory. Also, we note that the spaces $C^{\infty/2+n}(\boldsymbol{g}, V)$ with the representation defined in (2.12) are in the category $\mathcal{O}$.

We will denote the corresponding semi-infinite cohomology classes of ghost number n by $H^{\infty/2+n}(\boldsymbol{g}, V)$, or, sometimes, by $H^{\infty/2+n}(d, V)$.

Finally, given a subalgebra $\boldsymbol{h} \subset \boldsymbol{g}$ one defines a complex $C^{\infty/2+\bullet}(\boldsymbol{g}, \boldsymbol{h}; V)$ of semi-infinite cohomology relative to $\boldsymbol{h}$, as the subcomplex which consists of

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[1] In fact, this sole property may be used to restrict the modules in a more general approach to semi-infinite cohomology [33].

those chains in $C^{\infty/2+\bullet}(\boldsymbol{g}, V)$ which are annihilated by $b(x)$ and $\Pi(x)$ for all $x \in \boldsymbol{h}$.

This concludes our review of basic facts about the semi-infinite cohomology. For more detailed information and proofs of the results cited above the reader should consult one of the basic papers [10,11,28], and [33] for an abstract definition of semi-infinite cohomology as a derived functor of semi-invariants.

3. Basic computational techniques

As we will see in the following sections, physically relevant complexes typically come equipped with some additional structure. This is usually sufficient to allow an explicit computation of the cohomology using spectral sequences, a standard technique from homological algebra (see e.g. ref. [34,35]). In this section we will review such methods, keeping in mind their applications later on. Our discussion will proceed from the most general spectral sequences to the more specialized ones, which arise in the computation of Lie algebra cohomology. In the latter case most of the results will also be valid for the semi-infinite cohomology, and, unless some subtleties are present, we will simplify the notation and write n instead of $\infty/2 + n$.

3.1. COHOMOLOGY OF A FILTERED (GRADED) COMPLEX

Consider a complex $(\mathcal{C}, d)$ of complex vector spaces, where $\mathcal{C} = \bigoplus_n \mathcal{C}^n$ and the differential $d: \mathcal{C}^n \to \mathcal{C}^{n+1}$. Suppose that there is an additional gradation, such that for each order (ghost number) n [#2],

$$\mathcal{C}^n = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}^n_k. \tag{3.1}$$

We will refer to the integer k as the degree, and denote by π_k the projection onto the subspace of degree k. This gradation by the degree must satisfy the following properties:

(i) The differential d has only terms of nonnegative degree, i.e.

$$d = d_0 + d_1 + \cdots = d_0 + d_>, \tag{3.2}$$

where

$$d_i: \mathcal{C}^n_k \to \mathcal{C}^{n+1}_{k+i}. \tag{3.3}$$

(ii) In each order only a finite number of nontrivial degrees are present, i.e. for each n, spaces $\mathcal{C}^n_k$ are nontrivial for a finite number of k's.

[#2] This is stronger than the usual assumption that $\mathcal{C}$ must be a filtered complex. A standard filtration in our case, for which $\mathcal{C}$ is isomorphic with the graded object, is given by subspaces $F_p \mathcal{C}^n = \bigcup_{k \geq p} \mathcal{C}^n_k$.

The spectral sequence associated with such a gradation allows a systematic computation of the cohomology classes $H(d, \mathcal{C})$. It is a sequence of complexes $(E_r, \delta_r)_{r=0}^{\infty}$, such that

$$E_0 = \mathcal{C}, \quad E_1 = H(d_0, \mathcal{C}), \quad E_{r+1} \doteq H(\delta_r, E_r), \qquad r = 1, 2, \ldots . \quad (3.4)$$

The differentials are defined recursively, beginning for $r = 0$ with the definition $\delta_0 = d_0$. Now, for $r \geq 1$, notice that $\psi_k \in \mathcal{C}$ represents an element in E_r (possibly a trivial one) if and only if there exist $\psi_{k+1}, \ldots, \psi_{k+r-1}$ such that

$$\pi_i d(\psi_k + \psi_{k+1} + \ldots + \psi_{k+r-1}) = 0, \qquad \text{for } i = k, \ldots, k+r-1. \quad (3.5)$$

Then we may define $\delta_r \colon E_r \to E_r$ via

$$\delta_r \psi_k = \pi_{k+r} d(\psi_k + \psi_{k+1} + \ldots + \psi_{k+r-1}). \quad (3.6)$$

The E_1 term of this sequence is obviously well defined, because (3.2) together with $d^2 = 0$ imply that $d_0^2 = 0$. To verify that the subsequent terms are also well defined – in particular, that δ_r are nilpotent operators in E_r – becomes more and more tedious, so one usually resorts to more abstract techniques [34,35] rather than using the explicit formulae (3.5) and (3.6). Nevertheless, it is quite illuminating to work out by hand at least the next two terms (see, e.g. ref. [36]). It then becomes clear that the elements of E_r are those cohomology classes of d_0 which can be extended to approximate cohomology classes of d through r degrees.

A spectral sequence (E_r, δ_r) becomes a useful device provided it converges, which means that the spaces E_r stabilize, i.e.

$$E_r = E_{r+1} = \ldots = E_{\infty}, \quad (3.7)$$

for some $r \geq 1$. Obviously, this requires

$$\delta_r = \delta_{r+1} = \ldots = \delta_{\infty} = 0. \quad (3.8)$$

In such a case one also says that the sequence collapses at E_r.

As might be expected, one can prove the following fundamental theorem (see, e.g. ref. [34,35]):

Theorem 3.1. For a graded complex $(\mathcal{C}, d)$ as above

$$E_{\infty} \cong H(d, \mathcal{C}). \quad (3.9)$$

Most of spectral sequences, which we will encounter later in this paper, collapse at the first term, because E_1 turns out to be nontrivial only in either one ghost number or one degree (or both). As the differentials δ_r, $r \geq 1$, increase both the order and the degree, they must vanish, which implies $E_1 \cong E_{\infty}$, or, equivalently, $H(d_0, \mathcal{C}) \cong H(d, \mathcal{C})$. Clearly this happy circumstance will often be the result of

a clever choice for the definition of degree, which splits the calculation in this opportune way.

3.2. SPECTRAL SEQUENCES OF A DOUBLE COMPLEX

With a double complex $(\mathcal{C}, d, d')$

$$d: \mathcal{C}^{p,q} \to \mathcal{C}^{p+1,q}, \quad d': \mathcal{C}^{p,q} \to \mathcal{C}^{p,q+1}, \tag{3.10}$$

$$d^2 = d'^2 = dd' + d'd = 0, \tag{3.11}$$

one can associate two spectral sequences, (E_r, δ_r) and (E'_r, δ'_r), arising from the grading of the underlying single complex $(\mathcal{C}, D)$,.

$$\mathcal{C}^n = \bigoplus_{p+q=n} \mathcal{C}^{p,q}, \quad D = d + d', \tag{3.12}$$

by p and q, respectively. In the first case we have $\mathcal{C}_p = \oplus_q \mathcal{C}^{p,q}$, which gives

$$E_1 \cong H(d', \mathcal{C}) \quad \delta_1 = d, \tag{3.13}$$

while in the second case $\mathcal{C}'_q = \oplus_p \mathcal{C}^{p,q}$, and

$$E'_1 \cong H(d, \mathcal{C}) \quad \delta'_1 = d'. \tag{3.14}$$

One should note that the spaces E_r and E'_r in these spectral sequences are doubly graded, and the action of the induced differentials δ_r and δ'_r with respect to this gradation is

$$\delta_r: \qquad E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \tag{3.15}$$

$$\delta'_r: \qquad E_r'^{p,q} \longrightarrow E_r'^{p-r+1,q+r}. \tag{3.16}$$

In both cases the limit of the spectral sequence E_∞ or E'_∞, if it exists, yields the cohomology of the complex $\mathcal{C}$ with the differential $D = d + d'$. The following frequently used theorem summarizes in which sense the order of computing the cohomologies of d and d' can be interchanged.

Theorem 3.2. Suppose that in a double complex $(\mathcal{C}, d, d')$ both spectral sequences (E_r, δ_r) and (E'_r, δ'_r) collapse at the second term. Then

$$\bigoplus_{p+q=n} H^p(d, H^q(d', \mathcal{C})) \cong \bigoplus_{p+q=n} H^q(d', H^p(d, \mathcal{C})). \tag{3.17}$$

Proof: [34] We simply have

$$
\begin{aligned}
\bigoplus_{p+q=n} H^p(d, H^q(d', \mathcal{C})) \;&=\; \bigoplus_{p+q=n} E_2^{p,q} \\
&=\; H^n(D, \mathcal{C}) \\
&=\; \bigoplus_{p+q=n} E_2'^{p,q} \;=\; \bigoplus_{p+q=n} H^q(d', H^p(d, \mathcal{C})) \,.
\end{aligned}
\tag{3.18}
$$

3.3. "SPLIT AND FLIP" SPECTRAL SEQUENCES

Throughout this subsection we assume that g is either $\mathfrak{g}$, or $\widehat{\mathfrak{g}}$ or Vir or a subalgebra of those, and that all modules are highest weight modules in the category $\mathcal{O}$. We recall that in particular this means that any module V is also a module of $\mathfrak{t}, \widehat{\mathfrak{t}}$ or Vir_0, and there is a single state v_A, with the highest weight A, and the weight space decomposition of V is $V = \bigoplus_{\lambda \in P(V)} V_\lambda$, where $P(V) \subset D(A)$.

Of particular interest is the specific case in which V is the tensor product of two highest weight modules V_1 and V_2, with highest weights A_1 and A_2, respectively. We show here how to construct a spectral sequence that allows an estimate of the relative cohomology of the tensor product module $V_1 \otimes V_2$, in terms of the cohomologies of V_1 and V_2. This will be achieved by introducing a family of degrees on the complex $\mathcal{C}(g, V_1 \otimes V_2)$, which are a natural generalization of the f-degree defined in refs. [11,28].

Let f be an arbitrary integer valued function on the root lattice, such that [3]

$$
f(\alpha_i) \neq 0, \quad f(n_1\alpha_i + n_2\alpha_j) = n_1 f(\alpha_i) + n_2 f(\alpha_j),
\tag{3.19}
$$

for all simple roots α_i, α_j, and integers n_1 and n_2. Examples of such functions, which we will consider in the next section, are obtained by taking (for $g = \mathfrak{g}$)

$$
f(\alpha) = (\rho, w\alpha), \quad \alpha \in \Delta, \; w \in W,
\tag{3.20}
$$

or similarly (for $g = \widehat{\mathfrak{g}}$)

$$
f(\alpha) = (\widehat{\rho}, w\alpha), \quad \alpha \in \widehat{\Delta}, \; w \in \widehat{W}.
\tag{3.21}
$$

We will denote such f's by f_w.

Obviously, using f, we can decompose g into a direct sum

$$
g = n_-^f \oplus t \oplus n_+^f,
\tag{3.22}
$$

where n_+^f and n_-^f are subalgebras corresponding to the positive and negative roots defined with respect to f. We may also extend f to a highest weight

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[3] Here, and in the remainder of this section, we use the same notation for the roots and weights of all three algebras.

module V (with highest weight Λ) by setting

$$f(v) = f(\lambda - \Lambda), \quad \text{for } v \in V_\lambda. \tag{3.23}$$

Similarly, we can use the weight space decomposition to extend f to the ghost Fock space, $\mathcal{F}_{\text{gh}}$, by setting $f(\omega_0) = 0$. Note that in this case the ghosts and anti-ghosts corresponding to t will not change the value of f.

Let us concentrate on the relative cohomology of g in $V_1 \otimes V_2$. One of the effects of passing to the relative cohomology is that we drop the ghosts and anti-ghosts of t, i.e. $\mathcal{F}_{\text{gh,rel}} \cong \bigwedge(n_-^f \oplus n_+^f)^*$. This space decomposes into a tensor product $\mathcal{F}_{\text{gh,rel}} = \mathcal{F}_{\text{gh},-}^f \otimes \mathcal{F}_{\text{gh},+}^f \cong \bigwedge(n_-^f)^* \otimes \bigwedge(n_+^f)^*$. Combining that with the tensor product structure of the module itself, we are led to consider the following decomposition of the spaces in the complex:

$$C^n(g, t; V_1 \otimes V_2) = \bigoplus_{p+q=n} \bigoplus_\lambda C_\lambda^p(n_-^f, V_1) \otimes C_{-\lambda}^q(n_+^f, V_2), \tag{3.24}$$

where $C_\lambda(\cdot, \cdot)$ denotes the subspace with the weight λ. Note that because $C^n(\cdot, \cdot)$ is a module in category $\mathcal{O}$, the sum in (3.24) is in fact finite both with respect to n and λ. (This is clear for $g = \mathbf{g}$, and for $\widehat{\mathbf{g}}$ and Vir becomes obvious if we remember that all ghost and anti-ghost creation operators have positive energy, which is a part of the affine weight.) We emphasize that at this point (3.24) is merely an equality between *vector spaces*, and that the assignment of n_-^f to V_1 and n_+^f to V_2 is completely arbitrary.

However, it is easy to see that the differential d in $C(g, t; V_1 \otimes V_2)$ is of the form $d = d_- + d_+ + \cdots$, where d_- and d_+ are the differentials in $C_\lambda(n_-^f, V_1)$ and $C_{-\lambda}(n_+^f, V_2)$, respectively, while the additional terms correspond to the action of n_-^f and n_+^f on V_2 and V_1, respectively, and three-ghost terms that arise when n_-^f and n_+^f do not commute. This form of the differential suggests the introduction of a degree so that, at the first term of the spectral sequence, (3.24) becomes an equality between *complexes*. Such a degree, which we denote f deg, may be defined via

$$f \deg((v_1 \otimes \omega_1) \otimes (v_2 \otimes \omega_2)) = f(v_1) + f(\omega_1) - f(v_2) - f(\omega_2), \tag{3.25}$$

where $v_1 \in V_1$, $v_2 \in V_2$, $\omega_1 \in \mathcal{F}_{\text{gh},-}^f$ and $\omega_2 \in \mathcal{F}_{\text{gh},+}^f$. Note that on the ghosts,

$$f \deg\left(\prod_\alpha c^\alpha \prod_\beta b_\beta \, \omega_0\right) = \sum_\alpha |f(\alpha)| - \sum_\beta |f(\beta)|. \tag{3.26}$$

In other words each ghost c^α increases f by $|f(\alpha)|$, while b_β decreases f by $|f(\beta)|$. By virtue of the triangle inequality satisfied by $|f(\cdot)|$ applied to the three-ghost terms, and recalling that $\Pi(t) \equiv 0$ on the subcomplex of relative cohomology, it is easy to check that the f deg $= 0$ term of the differential in the complex on the l.h.s. is indeed the sum of the differentials in complexes on the r.h.s. of (3.24), i.e. $d_0 = d_+ + d_-$! As we have argued above, this filtration

is finite, and so the corresponding spectral sequence converges. Thus we have shown the following theorem.

Theorem 3.3. There exists a spectral sequence (E_r, δ_r) such that

$$E_1^n \cong \bigoplus_{p+q=n} \bigoplus_{\lambda} H_\lambda^p(\mathbf{n}_-^f, V_1) \otimes H_{-\lambda}^q(\mathbf{n}_+^f, V_2), \tag{3.27}$$

$$E_\infty^n \cong H^n(\mathbf{g}, \mathbf{t}; V_1 \otimes V_2). \tag{3.28}$$

For obvious reasons we will refer to this sequence as the "split and flip" spectral sequence.

3.4. REDUCTION THEOREMS

Let us now consider the situation when the cohomology of one of the complexes on the r.h.s. of (3.24) is particularly simple; namely, it has only one state, at ghost number number zero, e.g.

$$H^q(\mathbf{n}_+^f, V_2) = \delta^{q,0} \mathbb{C}_{-\lambda}. \tag{3.29}$$

Then, for each n, the sum in (3.27) collapses to just one term with $p = n$. In fact, as we will now show, the entire spectral sequence collapses, and we obtain the following reduction theorem.

Theorem 3.4. For a module V_2 satisfying (3.29) the spectral sequence of theorem 3.3 collapses at E_1 and we obtain

$$H^n(\mathbf{g}, \mathbf{t}; V_1 \otimes V_2) \cong H_\lambda^n(\mathbf{n}_-^f, V_1) \otimes \mathbb{C}_{-\lambda}. \tag{3.30}$$

Proof: We observe that in the case of relative cohomology, there is an additional relation between the f deg of the two factors in (3.25), namely

$$[f(v_1) + f(\omega_1)] + [f(v_2) + f(\omega_2)] = -f(\Lambda_1 + \Lambda_2) = \text{const.} \tag{3.31}$$

This shows that E_1^n can be nontrivial in only one f-degree – the same one for all n. Thus the sequence must collapse, as we have discussed at the end of section 3.1. $\qquad\square$

For $f = -f_{w=1} = -\deg$ this theorem becomes the reduction theorem of ref. [28].

3.5. RESOLUTIONS

A direct calculation of the cohomology on a given module will usually be complicated, and yet one can often find special modules for which the calculation is immediate. For an arbitrary g-module, V, a promising – and usually necessary – line of attack is to find a description of V in terms of these special modules. In fact, the possibility of carrying out such a procedure is a good test as to whether one is working in a sensible category of modules! A decomposition of V into its "building blocks" is achieved in terms of a resolution.

Definition 3.1. We say that a complex $(\mathcal{R}, \delta)$ of g-modules $\mathcal{R}^n$ and $\delta\colon \mathcal{R}^n \to \mathcal{R}^{n+1}$ intertwining with the action of g is a resolution of the g-module V if

$$H^n(\delta, \mathcal{R}) \cong \delta^{n,0} V. \tag{3.32}$$

Resolutions are said to be one-sided if we can set $n \geq 0$, or two-sided if $n \in \mathbb{Z}$, finite or infinite, etc.

A typical application is as follows. Suppose we want to compute $H(g, V)$, and we know how to write down a resolution of V in which $H(g, \mathcal{R}^n)$ are computable and simple. By replacing V with its resolution we have

$$H^n(g, V) = H^n(g, H^0(\delta, \mathcal{R})). \tag{3.33}$$

Clearly the form of the r.h.s. suggests that we consider the double complex $(\mathcal{C}, \delta, d)$, $\mathcal{C}^{n,m} = \mathcal{R}^n \otimes \bigwedge^m g^*$, and see whether theorem 3.2 can be applied to change the order of cohomologies in (3.33). If so, then we may exploit the simple structure of $H(g, \mathcal{R}^n)$. Indeed, in all the cases we discuss later on this is the best way to proceed.

4. Resolutions of highest weight modules of affine Kac–Moody algebras

In this section we review two classes of resolutions of irreducible highest weight modules of affine (untwisted) Kac–Moody algebra $\widehat{g}$: a one-sided resolution in terms of Verma modules, and a set of two sided resolutions in terms of Wakimoto modules. Similar constructions for the Virasoro algebra will be briefly summarized in section 6.

4.1. VERMA MODULES AND THE BGG-RESOLUTION

Recall that a Verma module M_Λ of $\widehat{g}$, with highest weight Λ, is freely generated by $\widehat{n}_-$ from a highest weight state v_Λ such that $n v_\Lambda = 0$, $n \in \widehat{n}_+$ and $t v_\Lambda = \Lambda(t) v_\Lambda$, $t \in \widehat{t}$. We will denote by M_Λ^* the contragradient Verma module [37], which is dual to M_Λ with respect to the canonical pairing in $\mathcal{U}(\widehat{g})$, and co-free

with respect to $\widehat{\mathbf{n}}_+$. It turns out that both of these modules can also be defined in a rather surprising manner [32,10].

Theorem 4.1. A Verma module M_Λ of $\widehat{\mathbf{g}}$ is completely characterized by the following two conditions:

- M_Λ is a module in category $\mathcal{O}$.
- As a $\widehat{\mathfrak{t}}$-module,

$$H^{\infty/2+n}(\widehat{\mathbf{n}}_-, M_\Lambda) \cong \delta^{n,0}\,\mathbb{C}_\Lambda. \tag{4.1}$$

By Poincaré duality in semi-infinite cohomology [11,28], a similar theorem holds for the contragradient Verma module $M_\Lambda^\star$, except that one must replace $\widehat{\mathbf{n}}_-$ with $\widehat{\mathbf{n}}_+$.

Proof: We sketch the proof [32]. To calculate (4.1), introduce an (increasing) filtration on the complex $C^{\infty/2+\bullet}(\widehat{n}_-, M_\Lambda)$ defined, on the basis of monomials, by

$$F^p C = \{v = e_{-\alpha_1}\cdots e_{-\alpha_r} b_{-\beta_1}\cdots b_{-\beta_s} v_\Lambda \mid r+s \le p\}. \tag{4.2}$$

Using the fact that M_Λ is freely generated by $\widehat{n}_-$ one finds that the first term in the associated spectral sequence is the so-called Koszul complex [29,35], for which it is easy to show that the cohomology is one-dimensional and generated by v_Λ. The spectral sequence then collapses to give (4.1). Note that one uses here only $\widehat{\mathbf{n}}_- \otimes \widehat{\mathfrak{t}}$-module structure of M_Λ!

As for the opposite part of the theorem, let V be a module in the category $\mathcal{O}$ whose cohomology is the same as that of M_Λ. Since in (4.1) $n \le 0$, we have $H^{\infty/2+0}(\widehat{\mathbf{n}}_-, V) \cong V/\widehat{\mathbf{n}}_- V$. (Note that such a characterization would not be valid if $n > 0$ terms were present in the complex!) Thus the module V is generated by $\mathcal{U}(\widehat{\mathbf{n}}_-)$ acting on the highest weight state v'_Λ, corresponding to the nontrivial 0th cohomology. Equivalently, we have a surjection σ from the Verma module M_Λ onto V, defined by $\sigma(v_\Lambda) = v'_\Lambda$. This gives a short exact sequence $0 \to K \to M_\Lambda \to V \to 0$, where $K = \ker \sigma$. From the corresponding long exact sequence in cohomology [35,34] we find that the cohomology of K must be trivial. This implies that $K = 0$, and thus $V \cong M_\Lambda$, as otherwise we would have a nontrivial module in the category $\mathcal{O}$ with zero cohomology, which is clearly impossible since it contains "highest" weights. $\square$

Before we formulate the next theorem, on the construction of resolutions in terms of Verma modules, we recall that for a given $w' \in \widehat{W}$, the length of w' is defined as $\ell(w') = |\Phi(w')|$, in terms of the set of roots $\Phi(w') = \widehat{\Delta}_+ \cap w'(\widehat{\Delta}_-)$ [31]. We also introduce a twisted action, denoted by $*$, of $\widehat{W}$ on the weights, $w' * \Lambda = \overline{w}'(\Lambda + \rho) - \rho + (k + h^\vee)\gamma$ for $w' = t_\gamma\overline{w}' \in \widehat{W}$, $\Lambda \in P^k$.

Theorem 4.2. Let $\Lambda \in P^k_+$ be an integrable weight of $\widehat{\mathbf{g}}$. Then there exists a resolution of the irreducible module L_Λ in terms of Verma modules given by the

complex $(\mathcal{M}_A^{(n)}, \delta^{(n)})$, $n \le 0$, such that

$$\mathcal{M}_A^{(n)} = \bigoplus_{\{w' \in \widehat{W} \,|\, \ell(w') = -n\}} M_{w' \star A}, \tag{4.3}$$

and the differentials $\delta^{(n)}$ are determined, up to combinatorial factors, by the singular vectors in M_A.

Proof: See ref. [32] for the proof for finite dimensional Lie algebras and refs. [38,39] for its generalization to Kac–Moody algebras.

4.2. WAKIMOTO MODULES AND FREE FIELD RESOLUTIONS

It is natural to ask whether any other $\widehat{\mathbf{g}}$-modules can be characterized in a way similar to the cohomological characterization of Verma modules and contragradient Verma modules, using subgroups isomorphic with $\widehat{\mathbf{n}}_-$ and/or $\widehat{\mathbf{n}}_+$. This question motivated the following construction of Wakimoto modules [40] proposed by Feigin and Frenkel some three years ago [23].

For a given $w \in W$, consider a subgroup $\widehat{\mathbf{n}}_+^w = w_\infty \cdot \widehat{\mathbf{n}}_+ \equiv w_\infty \widehat{\mathbf{n}}_+ w_\infty^{-1}$, where, formally, $w_\infty = \lim_{N \to \infty} w_N$, $w_N = w t_{N\rho} \in \widehat{W}$. The action of this infinite twist should be understood as

$$\widehat{\mathbf{n}}_+^w = \{x \in \widehat{\mathbf{g}} \,|\, \exists_{N_0} \forall_{N \ge N_0} \, x \in w_N \cdot \widehat{\mathbf{n}}_+\}. \tag{4.4}$$

Explicitly,

$$\widehat{\mathbf{n}}_+^w = \bigoplus_{\widehat{\alpha} \in \widehat{\Delta}_+^w} \widehat{\mathbf{g}}_{\widehat{\alpha}}, \tag{4.5}$$

where

$$\widehat{\Delta}_+^w = \{\widehat{\alpha} = n\delta + w\alpha \,|\, \alpha \in \Delta_+, n \in \mathbb{Z}\} \cup \{\widehat{\alpha} = n\delta \,|\, n > 0\}, \tag{4.6}$$

i.e. $\widehat{\mathbf{n}}_+^w$ is a sum of the current algebra based on $\mathbf{n}_+^w = w\mathbf{n}_+ w^{-1}$ (a twisted nilpotent subalgebra of $\mathbf{g}$), and the positive modes of currents corresponding to the Cartan subalgebra $\mathbf{t}$. An equivalent way of characterizing $\widehat{\Delta}_+^w$ is as those roots for which (see section 3.3) $f_{w_N}(\widehat{\alpha}) = (\widehat{\rho}, w_N \widehat{\alpha})$ is positive for N sufficiently large.

Let us also introduce

$$\widehat{\Delta}_+^{w,(\pm)} = \widehat{\Delta}_+^w \cap \widehat{\Delta}_\pm, \quad \widehat{\Delta}_-^{w,(\pm)} = (-\widehat{\Delta}_+^w) \cap \widehat{\Delta}_\pm, \tag{4.7}$$

and denote the corresponding subgroups by $\widehat{\mathbf{n}}_+^{w,(\pm)}$ and $\widehat{\mathbf{n}}_-^{w,(\pm)}$, respectively. We then have decompositions $\widehat{\mathbf{g}} = \widehat{\mathbf{n}}_-^w \oplus \widehat{\mathbf{t}} \oplus \widehat{\mathbf{n}}_+^w$ and $\widehat{\mathbf{n}}_\pm^w = \widehat{\mathbf{n}}_\pm^{w,(-)} \oplus \widehat{\mathbf{n}}_\pm^{w,(+)}$. Also, in analogy with the usual length ℓ above, one can introduce a "twisted length" ℓ_w on $\widehat{W}$, defined by [23,25]

$$\ell_w(w') = |\Phi^{w,(+)}(w')| - |\Phi^{w,(-)}(w')|, \quad \Phi^{w,(\pm)}(w') = \widehat{\Delta}_+^{w,(\pm)} \cap w'(\widehat{\Delta}_-). \tag{4.8}$$

With this somewhat elaborate machinery we have [23]

Definition 4.1. For $w \in W$, a Wakimoto module F_Λ^w of an affine Kac–Moody algebra $\widehat{\mathbf{g}}$ is a module such that
- F_Λ^w is a $\widehat{\mathbf{g}}$-module in category $\mathcal{O}$.
- As a $\widehat{\mathbf{t}}$-module

$$H^{\infty/2+n}(\widehat{\mathbf{n}}_+^w, F_\Lambda^w) \cong \delta^{n,0} \mathbb{C}_\Lambda . \tag{4.9}$$

To show that this definition is not vacuous, one must construct examples of such modules. In conformal field theory one can give an explicit realization of Wakimoto modules as Fock spaces of a set of conjugate first order bosonic free fields of conformal dimension $(1,0)$ (one such pair for every root $\alpha \in \Delta_+^w$), and a set of free scalar fields with background charge (as many as the rank of $\mathbf{g}$) (see e.g. ref. [41] and the references therein). The most frequently used realization appears to be for $w = 1$, but, as we will see in the next section, the ability to implement the general twist plays a crucial role in the construction of a resolution for the coset module.

Although free field realizations of the Wakimoto modules have been discussed at some length in the literature, their cohomology has only been considered in ref. [23]. Let us briefly outline here a proof of (4.9).

As with Verma modules, the Wakimoto modules F_Λ^w constructed thus far are particularly simple when viewed as $\widehat{\mathbf{n}}_+^w \oplus \widehat{\mathbf{t}}$ modules, namely,

$$F_\Lambda^w \cong \mathcal{U}(\widehat{\mathbf{n}}_+^{w,(-)}) \otimes \mathcal{U}(\widehat{\mathbf{n}}_-^{w,(-)})^\star \otimes \mathbb{C}_\Lambda . \tag{4.10}$$

We identify here $\mathcal{U}(\widehat{\mathbf{n}}_+^{w,(-)})$ with the Verma module of $\widehat{\mathbf{n}}_+^w$ built on the vacuum annihilated by the generators in $\widehat{\mathbf{n}}_+^{w,(+)}$. Similarly we define the contragradient module of $\widehat{\mathbf{n}}_+^w$ in the second factor. Finally, $\widehat{\mathbf{n}}_+^w$ acts trivially on the third factor, which is introduced to shift the highest weight to the desired value. In this explicit realization the cohomology computation is almost trivial. Consider an f deg as in section 3.3 with $f = f_\infty = \lim_{N\to\infty} f_{w_N}$. Identifying $\mathcal{U}(\widehat{\mathbf{n}}_+^{w,(-)})$ with V_1 and $\mathcal{U}(\widehat{\mathbf{n}}_-^{w,(-)})^\star$ with V_2, the first term of the resulting "split and flip" spectral sequence is given by

$$E_1 \cong \bigoplus_{n \leq 0} \bigoplus_{m \geq 0} H^{\infty/2+n}(\widehat{\mathbf{n}}_+^{w,(-)}, \mathcal{U}(\widehat{\mathbf{n}}_+^{w,(-)})) \otimes H^{\infty/2+m}(\widehat{\mathbf{n}}_+^{w,(+)}, \mathcal{U}(\widehat{\mathbf{n}}_-^{w,(-)})^\star) ,$$

$$\tag{4.11}$$

and, by virtue of theorem 4.1, it is nontrivial only for $m = n = 0$. Thus the sequence collapses, and we obtain (4.9).

There remains the intriguing mathematical question as to whether Wakimoto modules are uniquely determined by their cohomology. It is not too difficult to show that the cohomology (4.9) forces the module to be free over $\widehat{\mathbf{n}}_+^{w,(-)}$, and co-free over $\widehat{\mathbf{n}}_+^{w,(+)}$, which of course is manifest in the explicit realization (4.10).

To prove this one can consider two Hochschild–Serre spectral sequences [#4] [42] corresponding to two subgroups of $\hat{\mathbf{n}}_+^w$. Specifically, one easily sees that the differential d in the complex $\mathcal{C}(\hat{\mathbf{n}}_+^w, F)$ can be split into a sum of two anticommuting differentials, $d = d_+ + d_-$, corresponding to $\mathbf{n}_+^{w,(+)}$ and $\mathbf{n}_+^{w,(-)}$. Thus we have a structure of a double complex. The two Hochschild–Serre spectral sequences are then simply those arising via the usual bi-grading of this double complex, as discussed in section 3.2. After some rather subtle analysis, the freeness and co-freeness essentially follow from theorem 4.1. However, we were not able to complete the argument for the uniqueness of the module, and we are not aware of any existing explicit proof of this fact.

As expected we have the following resolutions [23,25].

Theorem 4.3. For any $\Lambda \in P_+^k$ and $w \in W$, there exists a resolution of L_Λ in terms of Wakimoto modules given by a complex $(\mathcal{F}_\Lambda^{w,(n)}, \delta^{w,(n)})$, where

$$\mathcal{F}_\Lambda^{w,(n)} = \bigoplus_{\{w' \in \widehat{W}\,;\,\ell_w(w') = n\}} F_{w'*\Lambda}^w, \quad n \in \mathbb{Z}. \tag{4.12}$$

In many applications in conformal field theory and topological field theories considering only integrable weights is not sufficient. Rather, we have to consider fractional levels, say

$$k + \mathrm{h}^\vee = \frac{p}{p'}, \quad \gcd(p,p') = 1, \quad \gcd(p',r^\vee) = 1, \quad p \geq \mathrm{h}^\vee, \quad p' \geq \mathrm{h}, \tag{4.13}$$

and weights of the form

$$\Lambda = \Lambda^{(+)} - (k + \mathrm{h}^\vee)\Lambda^{(-)}, \tag{4.14}$$

with $\Lambda^{(+)} \in P_+^{p-\mathrm{h}^\vee}$ and $\Lambda^{(-)} \in P_+^{\vee p'-\mathrm{h}}$. Here, $r^\vee$ is the "dual tier number" of $\hat{\mathbf{g}}$. We recall that in the simply laced case $\mathrm{h} = \mathrm{h}^\vee$, $P_+ = P_+^\vee$ and $r^\vee = 1$. Weights of the form (4.14) are a subset of the class of so-called admissible weights [43].

It turns out that the entire discussion of the case of integrable representations can easily be extended to this more general class of admissible weights. One can construct a complex as above in which

$$\mathcal{F}_\Lambda^{w,(n)} = \bigoplus_{\{w' \in \widehat{W}\,|\,\ell_w(w') = n\}} \mathcal{F}_{w'*\Lambda^{(+)} - (k+\mathrm{h}^\vee)\Lambda^{(-)}}, \tag{4.15}$$

and the differentials are precisely the same as in the resolution for the integrable weight $\Lambda^{(+)}$ [26,27]. We expect that this complex provides a Fock space resolution of the irreducible module L_Λ. For $\widehat{\mathrm{sl}(2)}$ (and $w = 1$) this has been proved in ref. [24].

[#4] We thank E. Frenkel for suggesting this line of reasoning.

5. BRST structure of G/H coset theories

From the point of view of representation theory for affine Kac–Moody algebras, the problem of constructing a coset model can be formulated as follows:

Given a pair of algebras $(\widehat{\mathbf{g}}, \widehat{\mathbf{h}})$, $\widehat{\mathbf{h}} \subset \widehat{\mathbf{g}}$, and a corresponding pair of weights (Λ, λ) (not necessarily integrable), construct the coset module $L_{\Lambda,\lambda}^{G/H}$ determined by the decomposition of L_{Λ}^{G} into irreducible $\widehat{\mathbf{h}}$-modules [#5]

$$L_{\Lambda}^{G} \cong \bigoplus_{\lambda} L_{\Lambda,\lambda}^{G/H} \otimes L_{\lambda}^{H}. \tag{5.1}$$

Although it is possible to proceed in such generality (see e.g. ref. [44]), we will assume here that the embedding of $\widehat{\mathbf{h}}$ in $\widehat{\mathbf{g}}$ arises from a regular embedding of $\mathbf{h}$ in $\mathbf{g}$, so that one can identify the Cartan subalgebras of $\mathbf{h}$ and $\mathbf{g}$ and choose the set of positive roots of $\mathbf{h}$ to be a subset of those in $\mathbf{g}$. Also, for regular embeddings, the Dynkin embedding index of $\mathbf{h}$ in $\mathbf{g}$ is $j = 1$. It is then clear that any highest weight $\widehat{\mathbf{g}}$-module in the category $\mathcal{O}$ is at least locally $\widehat{\mathbf{n}}_{+}^{H}$ finite as an $\widehat{\mathbf{h}}$-module (in cases which are relevant for applications these modules are in fact in the category $\mathcal{O}$ of $\widehat{\mathbf{h}}$-modules).

Recall that for the standard G/H coset models [13], which will be our main interest here, $\Lambda \in P_{+}^{G,k}$, while the sum in (5.1) runs over $\lambda \in P_{+}^{H,k}$. In that case we may in fact identify $L_{\Lambda,\lambda}$ with the set of $\widehat{\mathbf{h}}$-singular states in L_{Λ}^{G} at weight λ, and write down in terms of (semi-infinite) cohomology

$$L_{\Lambda,\lambda} \cong H_{\lambda}^{\infty/2+0}(\widehat{\mathbf{n}}_{+}^{H}, L_{\Lambda}^{G}). \tag{5.2}$$

(Note that for $\widehat{\mathbf{n}}_{+}^{H}$ the ordinary and the semi-infinite cohomologies coincide.)

However mathematically interesting, the complex $C_{\lambda}^{\infty/2+\bullet}(\widehat{\mathbf{n}}_{+}^{H}, L_{\Lambda}^{G})$ is not very attractive for a physicist as it only has states with positive ghost numbers, and thus it cannot arise in a covariant manner from any conformal field theory. Instead, the complexes we want to concentrate on can be derived in the framework of the BRST quantization of gauged WZW-models [16], and correspond to the constraints $\widehat{\mathbf{h}} \sim 0$ on a larger module $L_{\Lambda}^{G} \otimes V_{\lambda'}$, where $V_{\lambda'}$ is a suitable highest weight $\widehat{\mathbf{h}}$-module with highest weight λ' determined in terms of λ. Then the space of physical states is computed as the cohomology classes

$$H^{\infty/2+n}(\widehat{\mathbf{h}}, L_{\Lambda}^{G} \otimes V_{\lambda'}), \quad n \in \mathbb{Z}. \tag{5.3}$$

The level k' of the affine weight λ' must be $k' = -k - 2h^{\vee}$ (where $h^{\vee}$ is the dual Coxeter number of $\mathbf{h}$), so that the total central charge of $\widehat{\mathbf{h}}$ – including the ghosts contribution – vanishes, and thus the BRST operator is nilpotent [16]. The problem is to determine the appropriate module $V_{\lambda'}$, and to compute the

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[#5] Clearly one may often leave off the superscripts G, H, and G/H without causing confusion. We will do that whenever possible, and conversely, where it is essential we will include them.

resulting cohomology. For ease of presentation we restrict our attention to the relative cohomology. The absolute cohomology follows in the usual way [11].

5.1. COHOMOLOGY OF THE BRST COMPLEX OF THE COSET MODELS

For a given coset model, the BRST operator corresponding to the differential of the complex which computes (5.3) may be written explicitly in terms of the currents as [16]

$$d =: \oint \sum_{a=1}^{\dim h} c^a(z)[J_a(z) + \tfrac{1}{2}J_a^{\text{gh}}(z)] :, \tag{5.4}$$

where the normal ordering is with respect to the SL(2,$\mathbb{R}$) vacuum, and the J_a are $\widehat{h}$-currents on the product module. Since the ghost and anti-ghost fields have conformal weights 0 and 1, respectively, the physical vacuum ω_0 differs from the SL(2,$\mathbb{R}$) vacuum $|0\rangle_{\text{gh}}$ in the ghost sector by the zero modes of the $c^a(z)$ in n_+^H, and thus the weight of ω_0 is equal to $\sum_{\alpha \in \Delta_+^H} \alpha = 2\rho^H$.

From the decomposition (5.1) it is clear that

$$H^{\infty/2+\bullet}(\widehat{h},\widehat{t}; L_\Lambda^G \otimes V_{\lambda'}) \cong \bigoplus_\lambda L_{\Lambda,\lambda} \otimes H^{\infty/2+\bullet}(\widehat{h},\widehat{t}; L_\lambda^H \otimes V_{\lambda'}). \tag{5.5}$$

Thus one only needs to choose an appropriate $\widehat{h}$-module $V_{\lambda'}$, and compute the cohomology of $L_\lambda \otimes V_{\lambda'}$ for irreducible $\widehat{h}$-modules L_λ. In particular one would like to understand whether for some $V_{\lambda'}$ the sum on the r.h.s. collapses to just one term, as in such a case the l.h.s. would give us a cohomological description of the coset module which, unlike (5.2), is "covariant" with respect to the subalgebra $\widehat{h}$. Recalling the reduction theorem in section 3.3, and the cohomologies (4.1) and (4.9), it is natural to seek such $V_{\lambda'}$'s among Wakimoto modules and/or Verma modules. Obviously this choice is also natural from the point of view of the gauged WZW-model.

We will now proceed to compute (5.5) for $V_{\lambda'} = F_{\lambda'}^w$ and $M_{\lambda'}$. First a preparatory result, which is an obvious consequence of theorem 3.4 and the cohomologies (4.1) and (4.9) above.

Lemma 5.1. For a tensor product, $F_\lambda^w \otimes F_{\lambda'}^{ww_0}$ (w_0 is the longest element in the Weyl group, W^H), of two oppositely twisted Wakimoto $\widehat{h}$-modules with arbitrary weights λ and λ',

$$H^{\infty/2+n}(\widehat{h},\widehat{t}; F_\lambda^w \otimes F_{\lambda'}^{ww_0}) \cong \delta^{n,0}\delta_{-\lambda-2\rho,\lambda'} \, \mathbb{C}. \tag{5.6}$$

Similarly, for the Verma modules we have

$$H^{\infty/2+n}(\widehat{h},\widehat{t}; M_\lambda^\star \otimes M_{\lambda'}) \cong \delta^{n,0}\delta_{-\lambda-2\rho,\lambda'} \, \mathbb{C}. \tag{5.7}$$

(For $\widehat{\mathbf{h}} = \mathrm{sl}(N)$ and $w = 1$, the above result was already proved in refs. [18–22] using explicit realizations of both Wakimoto modules. One should note that in the present derivation one only uses cohomologies of separate Wakimoto modules with respect to complementary subgroups of $\widehat{\mathbf{h}}$.) By choosing a suitable resolution of the irreducible module L_λ^H we can now construct a double complex, as discussed in section 3.5, and using the above lemma essentially read off by inspection the following results.

Theorem 5.2. Let λ be an integrable weight of the Kac–Moody algebra $\widehat{\mathbf{h}}$ at level k and $\widehat{\mathbf{n}}_+^w$ a twisted subalgebra of $\widehat{\mathbf{h}}$ corresponding to a fixed $w \in W^H$. Then
  – As a $\widehat{\mathbf{t}}$-module

$$H^{\infty/2+n}(\widehat{\mathbf{n}}_+^w, L_\lambda) \cong \bigoplus_{\{w' \in \widehat{W} \mid \ell_w(w') = n\}} \mathbb{C}_{w' * \lambda} . \tag{5.8}$$

  – The following projection holds:

$$H^{\infty/2+n}(\widehat{\mathbf{h}}, \widehat{\mathbf{t}}; L_\lambda \otimes F_{\lambda'}^{w w_0}) \cong \bigoplus_{\{w' \in \widehat{W} \mid \ell_w(w') = n\}} \delta_{w' * \lambda, -\lambda' - 2\rho}\, \mathbb{C} . \tag{5.9}$$

  – The same result holds for the Verma module $M_{\lambda'}$ (resp. contragradient Verma module $M_{\lambda'}^\star$) if one substitutes $\widehat{\mathbf{n}}_+^w \to \widehat{\mathbf{n}}_-$ (resp. $\widehat{\mathbf{n}}_+$) and $\ell_w(w') \to -\ell(w')$ (resp. $\ell(w')$) in (5.8) and (5.9).

We should note that the first part follows directly from the mere existence of a resolution of L_λ^H in terms of Wakimoto modules (see theorem 4.3), and their cohomology. It was first derived in ref. [23] as a generalization, to twisted subalgebras, of the Bott theorem (part 3 of the above theorem) for integrable representations of Kac–Moody algebras [38,45]. One may also observe that for each n the direct sum in (5.8) consists of one term for $\mathbf{h} = \mathrm{sl}(2)$, and is infinite for higher rank algebras. On the other hand the δ-function in (5.9) will always project out zero or one term, the latter when $-\lambda' - 2\rho$ is in the image of an integrable weight λ under the twisted action of some w'.

One can prove similar theorems in the case of admissible representations where one should use resolutions as in (4.15). We will leave this as an exercise for the reader. (The $\widehat{sl}(2)$ case has been discussed in ref. [21].)

Finally, by putting everything together we may summarize the general case of the BRST cohomology for G/H models in the following theorem:

Theorem 5.3. Consider a pair of Kac–Moody algebras $(\widehat{\mathbf{g}}, \widehat{\mathbf{h}})$, an integrable weight $\Lambda \in P_+^{G,k}$, and an arbitrary weight $\lambda' \in P^{H,k'}$, where $k' = -k - 2h^\vee$. Then

– $H^{\infty/2+n}(\widehat{\mathbf{h}}, \widehat{\mathbf{t}}; L^G_\Lambda \otimes F^{ww_0}_{\lambda'})$ is always zero or one-dimensional, and can be nontrivial for at most one n,

$$H^{\infty/2+n}(\widehat{\mathbf{h}}, \widehat{\mathbf{t}}; L^G_\Lambda \otimes F^{ww_0}_{\lambda'}) = \bigoplus_{\{w' \in \widehat{W} \mid \ell_w(w') = n\}} L_{\Lambda, -(w')^{-1}*(\lambda')-2\rho} . \quad (5.10)$$

– In particular, if we take $\lambda' = -\lambda - 2\rho$, where λ is an integrable weight, $\lambda \in P^{H,k}_+$, we have

$$H^{\infty/2+n}(\widehat{\mathbf{h}}, \widehat{\mathbf{t}}; L^G_\Lambda \otimes F^{ww_0}_{-\lambda-2\rho}) = \delta^{n,0} L_{\Lambda,\lambda} . \quad (5.11)$$

5.2. FREE FIELD RESOLUTIONS OF THE COSET MODELS

The last theorem above provides the crucial step towards the construction of a resolution of the coset module, $L^{G/H}_{\Lambda,\lambda'}$, in terms of more manageable modules – one must now simply replace the irreducible $\widehat{\mathbf{g}}$ module L^G_Λ in a controlled way. Here we will show how this is done, and examine the consequences. Let $(\mathcal{R}_\Lambda, \delta)$ be a resolution of L_Λ in terms of modules in the category $\mathcal{O}$, e.g. in terms of Verma modules as in theorem 4.2 or in terms of Wakimoto modules as in theorem 4.3. Consider the double complex $(\mathcal{C}_{\Lambda,\lambda}, d_{\mathrm{rel}}, d')$ with spaces

$$\mathcal{C}^{m,n}_{\Lambda,\lambda'} \cong \mathcal{R}^m_\Lambda \otimes F^{ww_0}_{\lambda'} \otimes \mathcal{F}^n_{\mathrm{gh,rel}} , \quad (5.12)$$

and with differentials $d = d_{\mathrm{rel}}$ (the BRST operator of the relative cohomology) and $d' = (-1)^{\mathrm{gh}} \delta$, where the prefactor counting the number of ghosts is introduced so that d and d' anticommute. Let $D = d + d'$ be the total differential in this complex.

Theorem 5.4. For a pair of Kac–Moody algebras $(\widehat{\mathbf{g}}, \widehat{\mathbf{h}})$ and a corresponding pair of integrable weights (Λ, λ), the cohomology of the complex $(\mathcal{C}_{\Lambda,-\lambda-2\rho}, D)$ defined above is

$$H^n(D, \mathcal{C}_{\Lambda,-\lambda-2\rho}) \cong \delta^{n,0} L_{\Lambda,\lambda} . \quad (5.13)$$

Proof: The proof follows from the first spectral sequence in section 3.2 and theorem 5.3 above – simply note that the E_1-term given in (3.13) has cohomology given by (5.11), and thus the spectral sequence collapses at E_2. $\quad\square$

In general one must be careful here in which order the cohomologies of d and d' are computed, as the other spectral sequence corresponding to this double complex may not collapse at the second term! It does, however, collapse if one chooses the resolution of L_Λ to be in terms of Wakimoto modules of $\widehat{\mathbf{g}}$ that, as $\widehat{\mathbf{h}}$-modules, are oppositely twisted to $F^{ww_0}_{\lambda'}$; i.e. the subalgebras $\widehat{\mathbf{n}}^{w(-)}_+$ and $\widehat{\mathbf{n}}^{(+)}_+$ of $\widehat{\mathbf{h}}$ act freely and cofreely, respectively. Thus, in this case one has further reduced the problem, so that the resolution is via a complex whose spaces are just the

cohomology spaces $H^{\infty/2+\bullet}(d, \mathcal{R}^n \otimes F_{\lambda'}^{ww_0})$, and whose differential is just that induced from δ on these spaces.

When the resolution of L_Λ^G is in terms of Wakimoto modules, we will call the complex of theorem 5.4 a free field resolution of the G/H coset model.

Of course there are other ways of splitting the differential D in this complex. In particular, one may try to eliminate the second Wakimoto module by using the reduction theorem. This can easily be achieved by introducing f_w-degree. The 0th order differential with respect to f_w-degree is $D_0 = d' + d_+ + d_-$, where $d_\pm$ are the BRST charges corresponding to $\hat{\mathbf{n}}_+^w$ and $\hat{\mathbf{n}}_-^w = \hat{\mathbf{n}}_+^{ww_0}$, respectively, precisely as in section 3.3. Using the reduction theorem we obtain then a smaller complex, $(\mathcal{C}_{(\mathrm{red})\Lambda,\lambda}, d_+, d')$, in which

$$\mathcal{C}_{(\mathrm{red})\Lambda,\lambda}^{m,n} \cong (\mathcal{R}_\Lambda^m \otimes \mathcal{F}_{\mathrm{gh},+}^n)_\lambda, \tag{5.14}$$

where $(\cdot)_\lambda$ denotes the projection onto the subspace with the weight λ. Obviously, the cohomology of this reduced complex also yields the same coset module.

If we carried out the same reduction procedure starting from a resolution of the coset module, but with $M_{\lambda'}$ rather than $F_{\lambda'}^{ww_0}$, we would arrive at the original "free field" realization of the coset model as constructed in [17].

In the latter case one may even further reduce the complex by evaluating the BRST cohomology of d_+. At ghost number equal to zero it clearly consists of the subspaces $\mathcal{S}_{\Lambda,\lambda}^m$ of singular vectors in $\mathcal{R}^m$. Although it is not clear whether the spectral sequence (whose first term we have just computed) converges at the second term [#6], one can give a separate (and rather subtle) argument which proves that the complex $(\mathcal{S}_{\Lambda,\lambda}, \delta)$ yields a resolution of the coset module. Note that this complex is just a naive extension of (5.2), and it is the change of order in which two cohomologies are evaluated that introduces the complication which must be dealt with separately. We refer the reader to ref. [17] and ref. [41] for a more complete discussion.

6. BRST cohomology of 2D gravity

In this section we will show how the results of the previous sections, in particular the reduction theorem, can be applied to compute physical states for 2D gravity coupled to $c \leq 1$ conformal matter. The relevant algebra in this case is $\mathbf{g} = \mathrm{Vir}$, while the interesting modules are the Virasoro Verma modules $M_{\Lambda,c}$, irreducible modules $L_{\Lambda,c}$ and the so-called Feigin–Fuchs (or Fock space) modules $F_{p,Q}$. Verma modules and irreducible modules are both labelled by their conformal dimension Λ, i.e. L_0 eigenvalue of the highest weight state, while the

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[#6] Note that we have changed the order in which the cohomologies of d_+ and d' are computed.

Feigin–Fuchs modules are labelled by the momentum p and background charge Q of the free scalar field (we refer to ref. [5] for any required clarification of the conventions). While the gravity sector will always be represented by a Feigin–Fuchs module F_{p^L,Q^L}, for the matter sector one has the choice of taking a Feigin–Fuchs module F_{p^M,Q^M} (gravity coupled to a free scalar field, i.e. the two-dimensional string), or an irreducible module L_{Δ^M,c^M} (gravity coupled to a minimal model).

The crucial difference with the affine Kac–Moody case is that, contrary to the Wakimoto modules, Feigin–Fuchs modules in general do not have a simple cohomology due to the fact that they are neither free nor co-free over part of the Virasoro algebra. As a consequence one finds that the cohomology $H(\text{Vir}, \text{Vir}_0; F_{p^M,Q^M} \otimes F_{p^L,Q^L})$ is already nontrivial, i.e. discrete states occur, contrary to what we have seen for the analogous cohomology in the affine Kac–Moody case. Nevertheless, some of the techniques of the previous sections can be applied by using specific properties of Feigin–Fuchs modules. For instance, by making use of the fact that $c \geq 25$ Fock modules $F_{p,Q}$ are isomorphic to either the Verma module $M_{\Delta,c}$ or the contragradient Verma module $M^*_{\Delta,c}$, depending on whether $\eta(p) = \text{sign}(i(p - Q))$ is positive or negative, respectively [46], we find by applying the reduction theorem

$$H^n(\text{Vir}, \text{Vir}_0; L_{\Delta^M,c^M} \otimes F_{p^L,Q^L}) \cong H^n(\text{Vir}_{\eta(p^L)}, L_{\Delta^M,c^M})_{1-\Delta(p^L)} \qquad (6.1)$$

In order to compute $H^n(\text{Vir}_{\eta(p^L)}, L_{\Delta^M,c^M})$, we take a resolution of L_{Δ^M,c^M} in terms of (contragradient) Verma modules [47], depending on $\eta(p^L)$, and proceed as outlined in section 3. The result can be summarized as follows [3–6,48]:

Theorem 6.1. Let $(\mathcal{R}^{(n)}, d')$ be a resolution of $L_{\Delta,c}$, where each $\mathcal{R}^{(n)}$ is a direct sum of Verma modules, contragradient Verma modules or Feigin–Fuchs modules. Let $\Delta^{(n)}$ be the conformal dimension of any one of the modules in $\mathcal{R}^{(n)}$. Denote $\widetilde{\mathcal{E}}(\Delta, c) = \{1 - \Delta^{(n)}\}$. For $\Delta = 1 - \Delta^{(n)}$, for some n, define $d(\Delta) = |n|$ (these definitions for $\widetilde{\mathcal{E}}(\Delta, c)$ and $d(\Delta)$ do not depend on the specific resolution). Then we have

(i) $H(\text{Vir}, \text{Vir}_0; L_{\Delta^M,c^M} \otimes F_{p^L,Q^L}) \neq 0$ iff $\Delta(p^L) \in \widetilde{\mathcal{E}}(\Delta^M, c^M)$.

(ii) For $\Delta(p^L) \in \widetilde{\mathcal{E}}(\Delta^M, c^M)$ we have

$$\dim H^n(\text{Vir}, \text{Vir}_0; L_{\Delta^M,c^M} \otimes F_{p^L,Q^L}) = \delta_{n,\eta(p^L)d(\Delta(p^L))}. \qquad (6.2)$$

7. Concluding remarks

We have shown in this paper how various methods of homological algebra can be applied to compute BRST cohomologies that arise in conformal field theory. The central role here is played by a reduction theorem which, roughly speaking, allows one to project out subspaces of a module by first tensoring it

with a suitable highest weight module and the computing the BRST cohomology of the product module. In particular, we have used this technique to construct free field resolutions of G/H coset theories, by imposing the BRST operator on the usual free field resolution of the WZW-model of G tensored with an appropriate Fock space of the free field realization of the WZW-model of H. These new resolutions of coset theories should allow one to repeat the same steps that have been carried out in the case of the usual (i.e. ungauged) WZW-models (see e.g. ref. [41] and the references therein). The most important outstanding problem in this direction is the construction of screened vertex operators, and the computation of the fusion rules.

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